FINITE TIME BLOW UP FOR CRITICAL WAVE EQUATIONS IN HIGH DIMENSIONS

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ABSTRACT. We prove that solutions to the critical wave equation (1.1) can not be global if the initial values are positive somewhere and nonnegative. This completes the solution to the famous blow up conjecture about critical semilinear wave equations of the form $\Delta u - \partial_t^2 u + |u|^p = 0$ in dimensions $n \geq 4$. The lower dimensional case $n \leq 3$ was settled many years earlier.

1. Introduction

Let $n \geq 2$ and $\Delta = \Sigma_{i=1}^n \partial^2/\partial x_i^2$ be the Laplace operator. We study the blow up of solutions to the following semilinear wave equation:

$$\begin{cases} \Delta u - \partial_t^2 u + |u|^p = 0 \quad \text{in} \quad \mathbf{R}^n \times (0,\infty), \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) \quad \text{in} \quad \mathbf{R}^n, \end{cases} \tag{1.1}$$

where the initial values satisfy

$$\begin{cases} (u_0, u_1) \in H^1(\mathbf{R}^n) \times L^2(\mathbf{R}^n), \\ u_0(x) = u_1(x) = 0 \quad \text{for} \quad |x| > R > 0, \end{cases}$$

and $p \in (1,\ p_c(n)]$. Here $p_c(n)$ is the positive root of the quadratic equation

$$(n-1)p^2 - (n+1)p - 2 = 0.$$

The number $p_c(n)$ is known as the critical exponent of problem (1.1), since it divides $(1,\ \infty)$ into two subintervals so that the following take place: If $p \in (1,\ p_c(n))$, then solutions with nonnegative initial values blow up in finite time; if $p \in (p_c(n),\ \infty)$, then solutions with small (and sufficiently regular) initial values exist for all time (see [St] e.g.). The proof has an interesting and exciting history that spans three decades. We only give a brief summary here and refer the reader to [St], [L], [DL] and a recent paper [JZ] for details. The problem about existence or nonexistence of global solutions is sometimes referred to as the conjecture of Strauss [St2]. The question was also asked by Glassey [G2] and Levine [L].

The case $n = 3$ was considered by John [J] who proved that nontrivial solutions must blow up in finite time when $1 < p < p_c(3)$. He also showed that global solutions exist for small initial values when $p > p_c(3)$. Glassey [G1], [G2] established the same results in the case $n = 2$. In [GLS] Georgiev, Lindblad and Sogge showed the existence of global solutions for small initial values when $p > p_c(n)$ and $n \geq 4$. (See also the work of Kubo and Kubota [KK], Lindblad and Sogge [LS] and Tataru[T]). The corresponding blow up result for $1 < p < p_c(n)$ and $n \geq 4$ was established by Sideris [Si]. Although the ideas of [Si] are very clear, the computations are quite sophisticated, involving spherical harmonics and other special functions. The proof was simplified by Ramaha [R] and Jiao and Zhou [JZ].

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The critical case $p = p_c(n)$ was studied by Schaeffer [Sc] in dimensions $n = 2$ and 3. Improving the lower bounds on the solution in [G2] and [Si], he was able to show that the critical powers belong to the corresponding blow up intervals. Despite the long effort, whether the critical powers $p_c(n)$ belong to the blow up intervals remains wide open in dimensions $n \geq 4$. The main obstruction to the method of [Sc] is that the Riemann function changes sign in high dimensions. This difficulty is not present if the initial values are large; the work of Levine [L] shows that such solutions blow up in finite time. Thus, the open problem is to show blow up without the latter assumption.

Here, we complete the solution of this conjecture about equation (1.1), thus filling the missing link since the 80s. Our main result is the following theorem.

Theorem 1.1. *Let u_0 and u_1 be non-negative and let either of them be positive somewhere. Suppose that problem (1.1) has a solution $(u, u_t) \in C([0,T),\ H^1(\mathbf{R}^n) \times L^2(\mathbf{R}^n))$ such that*

$$supp(u, u_t) \subset \{(x,t) : |x| \leq t + R\}.$$

If $p = p_c(n)$, then $T < \infty$.

We should mention that the existence of local in time solutions with the above regularity and support is well known. See p381 in [Si], for example. We prove Theorem 1.1 in Section 2. The cases $n = 2$ and 3 are proven in [Sc], so we concentrate on the case $n \geq 4$. Following the tradition, we consider $\int_{\mathbf{R}^n} u(x,t)dx$, where u is a local solution of problem (1.1). We show that this quantity satisfies a nonlinear differential inequality and, additionally, admits a lower bound $O(K(t)t^{n+1-(n-1)p/2})$ with $K(t) \geq \ln t$ as $t \to \infty$. The finite time blow up then follows immediately. Our estimate improves $K(t) \geq 1$, which is sufficient to show blow up only in the subcritical case. The new tools used are the Radon transform and the one-dimensional transform $\mathbf{T}$ (see (2.16)). These together with the L^p boundedness of the maximal function yield the extra $\ln t$ factor in our lower bound.

2. Proof of Theorem 1.1

The proof is carried out in several steps. We assume that $p = p_c(n)$ and $n \geq 4$.

Step 1.

We will need the following ODE result.

Lemma 2.1. *Let $p > 1$, $a \geq 1$, and $(p-1)a = q-2$. Suppose $F \in C^2([0,T))$ satisfies, when $t \geq T_0 > 0$,*

$$(a) \qquad F(t) \geq K_0(t+R)^a,$$

$$(b) \qquad \frac{d^2F(t)}{dt^2} \geq K_1(t+R)^{-q}[F(t)]^p,$$

with some positive constants K_0, K_1, T_0 and R. Fixing K_1, there exists a positive constant c_0, independent of R and T_0 such that if $K_0 \geq c_0$, then $T < \infty$.

Proof.

First let us make a translation $\tau = t - T_0$ and define $G = G(\tau) = F(\tau + R)$. Then for $\tau \geq 0$, one has

$$G(\tau) \geq K_0(\tau + T_0 + R)^a, \qquad \frac{d^2G(\tau)}{d\tau^2} \geq K_1(\tau + T_0 + R)^{-q}[G(\tau)]^p.$$

We take the change of variables $\tau = (T_0+R)s$ and $G_R = G_R(s) = (T_0+R)^{-a}G((T_0+R)s)$. Then easy computation shows that

$$G_R(s) \geq K_0(s+1)^a, \qquad \frac{d^2G_R(s)}{ds^2} \geq K_1(s+1)^{-q}[G_R(s)]^p,$$

when $s \geq 0$. Following the argument in [Si], p386, we know that G_R has to blow up in finite time if $K_0 \geq c_0$, which is sufficiently large. Clearly c_0 does not depend on R or T_0. Therefore F must also blow up in finite time. □

Step 2.

We introduce the function

$$\phi_1(x) = \int_{S^{n-1}} e^{x\cdot\omega}d\omega.$$

It is well known that

$$\phi_1(x) \sim C_n|x|^{-(n-1)/2}e^{|x|} \quad \text{as} \quad |x| \to \infty.$$

Suppose (1.1) has a global solution under the given initial values. Define

$$\begin{aligned} F_0(t) &= \int u(x,t)dx, \\ F_1(t) &= \int u(x,t)\psi_1(x,t)dx, \\ \psi_1(x,t) &= \phi_1(x)e^{-t}. \end{aligned} \tag{2.2}$$

To show that F_0 satisfies the differential inequality in Lemma 2.1 for suitable a, q, we integrate equation (1.1) over $\mathbf{R}^n$. We know that the support of $u(\cdot,t)$ is contained in $B(0,t+R)$ since the supports of u_0, u_1 are contained in $B(0,R)$. Hence using integration by parts, we obtain

$$\frac{d^2F_0(t)}{dt^2} = \int |u(x,t)|^p dx. \tag{2.2$'$}$$

Estimating the right side by the Hölder inequality, we have

$$\int |u(x,t)|^p dx \geq \frac{\left|\int u(x,t)dx\right|^p}{\left(\int_{|x|\leq t+R} dx\right)^{p-1}}.$$

Since

$$\int_{|y|\leq t+R} dx = \text{vol}\{x : |x| < t+R\} = \text{vol}(\mathbf{B}^n)(t+R)^n,$$

we obtain the differential inequality

$$\frac{d^2F_0(t)}{dt^2} \geq K_1(t+R)^{-n(p-1)}|F_0(t)|^p \tag{2.3}$$

with $K_1 = 1/(\text{vol}(\mathbf{B}^n))^{p-1}$.

To show that F_0 admits the lower bound in Lemma 2.1 (a), we relate d^2F_0/dt^2 to F_1 using again equation (1.1) and Hölder's inequality:

$$\frac{d^2F_0(t)}{dt^2} = \int |u(x,t)|^p dx \geq \frac{\left|\int u(x,t)\psi_1(x,t)dy\right|^p}{\left(\int_{|x|\leq t+R}[\psi_1(x,t)]^{p/(p-1)}dx\right)^{p-1}}.$$

By (2.2), the above becomes

$$\frac{d^2F_0(t)}{dt^2} \geq \frac{|F_1(t)|^p}{\left(\int_{|x|\leq t+R}[\psi_1(x,t)]^{p/(p-1)}dx\right)^{p-1}}. \tag{2.4}$$

In the following we estimate the denominator and numerator, respectively.

We claim that for all $t \geq 0$, $R > 0$,

$$I(t) \equiv \int_{|x|\leq t+R} [\psi_1(x,t)]^{p/(p-1)}dx \leq Ce^{p'R}(t+R)^{n-1-(n-1)p'/2}, \tag{2.5}$$

where $p' = p/(p-1)$. The claim is an immediate consequence of the observation

$$I(t) \leq C_1 e^{-p't} \int_0^{t+R} (1+r)^{-(n-1)p'/2} e^{p'r} r^{n-1} dr,$$

with $p' = p/(p-1)$ and integration by parts. Here we just used the formula

$$\psi_1(x,t) = e^{-t}\phi_1(x) \sim C_n|x|^{-(n-1)/2}e^{|x|-t} \quad \text{as} \quad |x| \to \infty.$$

Next we have

Lemma 2.2. *For all $t \geq 0$,*

$$F_1(t) \geq \frac{1}{2}(1-e^{-2t})\int [u_0(x)+u_1(x)]\phi_1(x)dx + e^{-2t}\int u_0(x)\phi_1(x)dx \geq c > 0.$$

Taking the lemma for granted, we combine it with (2.5) and with (2.4) to obtain

$$\frac{d^2F_0(t)}{dt^2} = \int_{\mathbf{R}^n} |u(x,t)|^p dx \geq C_0 L_2 (t+R)^{n-1-(n-1)p/2}, \quad t \geq 0, \tag{2.5$'$}$$

where

$$L_2 \geq \left(C\int u_0(x)\phi_1(x)dx\right)^p, \qquad C > 0. \tag{2.6}$$

Integrating twice, we have the estimate

$$F_0(t) \geq cL_2(t+R)^{n+1-(n-1)p/2} + \frac{dF_0(0)}{dt}t + F_0(0)$$

with some $c > 0$ depending only on n. When $p = p_c(n)$, it is easy to check that $n+1-(n-1)p/2 > 1$. Hence the following estimate is valid for all $t \geq 0$:

$$F_0(t) \geq K_0(t+R)^{n+1-(n-1)p/2}. \tag{2.7}$$

with $K_0 \equiv cL_2$. Here we remark that (2.7) have been proven in [Si] and [JZ] by different method. The current method, adopted from [YZ], seems much shorter.

If K_0 is sufficiently large, estimates (2.7), (2.3), and Lemma 2.1 with parameters

$$a \equiv n+1-(n-1)p/2 \quad \text{and} \quad q \equiv n(p-1)$$

would imply Theorem 1.1 since $p = p_c$ satisfies

$$(p-1)(n+1-(n-1)p/2) = n(p-1)-2 \quad \text{and} \quad p > 1.$$

However we have no control on the size of K_0. In the remainder of the paper, we will show that the lower bound (2.7) can be improved by a factor of $\ln t$ when t is large. Before doing so let us give a

Proof Lemma 2.2.
We multiply equation (1.1) by a test function $\psi \in C^2(\mathbf{R}^{n+1})$ and integrate over $\mathbf{R}^n \times [0,\, t]$.

$$\begin{aligned}\int_0^t \int u\ (\Delta\psi - \partial_s^2\psi) dyds + \int_0^t \int |u|^p \psi\, dyds \\ = \int (\partial_s u\ \psi - u\partial_s\psi) dy\Big|_{s=0}^{s=t}.\end{aligned} \tag{2.8}$$

We will apply this identity to $\psi = \psi_1$. Notice that for a fixed t, $u(\cdot, t) \in H_0^1(B(0, t+R))$. Hence all terms involving lateral boundary vanish during integration by parts. Notice also that

$$\partial_t \psi_1 = -\psi_1, \quad \Delta\psi_1 - \partial_t^2 \psi_1 = 0,$$

and

$$\begin{aligned}\int (\partial_t u\psi_1 - u\partial_t\psi_1) dy &= \int (\partial_t u\psi_1 + u\partial_t\psi_1) dy - 2\int u\partial_t\psi_1 dy \\ &= \frac{d}{dt}\int u\psi_1 dy + 2\int u\psi_1 dy.\end{aligned}$$

Hence, (2.8) becomes

$$\frac{dF_1(t)}{dt} + 2F_1(t) = \int [u(x,0) + \partial_t u(y,0)]\phi_1(y) dy + \int_0^t \int |u(y,s)|^p \psi_1(y,s) dyds.$$

Since $\psi_1 > 0$, we conclude that

$$\frac{dF_1(t)}{dt} + 2F_1(t) \geq \int [u(y,0) + \partial_t u(y,0)]\phi_1(y) dy.$$

We multiply by e^{2t} and integrate on $[0,\, t]$. Then

$$e^{2t}F_1(t) - F_1(0) \geq \frac{1}{2}(e^{2t} - 1)\int [u_0(y) + u_1(y)]\phi_1(y) dy.$$

Dividing through by e^{2t}, we obtain the lower bound in the Lemma. □

Step 3.

With no loss of generality we assume that $u(\cdot, t)$ is radial. This is so because one can use Daboux's identity to transform the problem into a suitable inequality in the radial case. i.e the sperical average of u, called $\bar{u}$ satisfies

$$\partial_t^2 \bar{u} - \Delta\bar{u} \geq |\bar{u}|^p.$$

Let $w \in \mathbf{R}^n$ be a unit vector. The Radon transform of u with respect to the space variables is defined as

$$\mathbf{R}(u)(\rho, t) = \int_{x\cdot w=\rho} u(x,t) dS_x, \tag{2.9}$$

where dS_x is the Lebesque measure on the hyper-plane $\{x \mid x \cdot w = 0\}$. Next we show that $\mathbf{R}(u)$ is a function of ρ and t and is in fact independent of w.

From (2.9) and the assumption that $u(\cdot, t)$ is radial, it is clear that

$$\begin{aligned}\mathbf{R}(u)(\rho, t) &= \int_{\{x' \ \mid \ x' \cdot w = 0\}} u(\rho w + x', t) dS_{x'} \\ &= c_n \int_0^\infty u(\sqrt{\rho^2 + |x'|^2}, t)|x'|^{n-2} d|x'|.\end{aligned}$$

Using the change of variable $r = \sqrt{\rho^2 + |x'|^2}$, we have

$$\mathbf{R}(u)(\rho, t) = c_n \int_{|\rho|}^\infty u(r, t)(r^2 - \rho^2)^{(n-3)/2} \ rdr. \tag{2.10}$$

This shows that $\mathbf{R}(u)(\rho, t)$ is independent of w. In the remainder of the step, we will derive a lower bound for $\mathbf{R}(u)(\rho, t)$.

Since u is a solution to (1.1), it is well known that $\mathbf{R}(u)$ satisfies the one dimensional wave equation

$$\partial_t^2 \mathbf{R}(u)(\rho, t) - \partial_\rho^2 \mathbf{R}(u)(\rho, t) = \mathbf{R}(|u|^p)(\rho, t). \tag{2.11}$$

From the D' Alembert's formula and the assumption that the initial values of u are nonnegative, one obtains

$$\mathbf{R}(u)(\rho, t) \geq \frac{1}{2} \int_0^t \int_{\rho-(t-s)}^{\rho+(t-s)} \mathbf{R}(|u|^p)(\rho_1, s) d\rho_1 ds. \tag{2.12}$$

Observe that the support of $u(\cdot, s)$ is contained in $B(0, s+R)$, the ball of radius R, centered at the origin. If $|\rho_1| > s + R$, then, for vectors y perpendicular to a unit vector w,

$$|\rho_1 w + y| = \sqrt{|\rho_1|^2 + |y|^2} \geq |\rho_1| > s + R.$$

Therefore

$$\mathbf{R}(|u|^p)(\rho_1, s) = \int_{\{y \ |y \cdot w = 0\}} |u(\rho_1 w + y, s)|^p dS_y = 0.$$

This shows that

$$supp \ \mathbf{R}(|u|^p)(\cdot, s) \subset B(0, s + R). \tag{2.13}$$

From now on we will assume $\rho \geq 0$, unless stated otherwise. If $s \leq (t - \rho - R)/2$, then

$$\rho + (t - s) \geq s + R, \qquad \rho - (t - s) \leq -(s + R).$$

Using this, (2.12) and (2.13), we deduce

$$\begin{aligned}\mathbf{R}(u)(\rho, t) &\geq \frac{1}{2} \int_0^{(t-\rho-R)/2} \int_{\rho-(t-s)}^{\rho+(t-s)} \mathbf{R}(|u|^p)(\rho_1, s) d\rho_1 ds \\ &= \frac{1}{2} \int_0^{(t-\rho-R)/2} \int_{-\infty}^{\infty} \mathbf{R}(|u|^p)(\rho_1, s) d\rho_1 ds \\ &= \frac{1}{2} \int_0^{(t-\rho-R)/2} \int_{\mathbf{R}^n} |u(y, s)|^p dyds.\end{aligned} \tag{2.14}$$

Recall from (2.5′) in step 2 that

$$\int_{\mathbf{R}^n} |u(y, s)|^p dy \geq cs^{(n-1)-(n-1)p/2}.$$

Note that $p \leq 2$ when $n \geq 4$. Therefore $(n - 1) - (n - 1)p/2 \geq 0$.

Substituting this to (2.14), we arrive that

$$\mathbf{R}(u)(\rho,t) \geq c(t-\rho-R)^{n-(n-1)p/2}, \qquad \rho \geq 0. \tag{2.15}$$

Step 4. For any function $f \in L^p(\mathbf{R})$, we introduce the transformation

$$\mathbf{T}(f)(\rho) = \frac{1}{|t-\rho+R|^{(n-1)/2}} \int_\rho^{t+R} f(r)|r-\rho|^{(n-3)/2} dr. \tag{2.16}$$

Observe that

$$\begin{aligned} |\mathbf{T}(f)(\rho)| &\leq \frac{1}{|t-\rho+R|}\left| \int_\rho^{t+R} |f(r)| dr \right| \\ &\leq \frac{2}{2|t-\rho+R|}\left| \int_{-(t+R)+2\rho}^{t+R} |f(r)| dr \right| \\ &\leq 2M(|f|)(\rho), \end{aligned}$$

where $M(|f|)$ is the maximal function of f. Therefore, there exists a $C > 0$, independent of t such that

$$\|\mathbf{T}(f)\|_p \leq C\|f\|_p. \tag{2.17}$$

Here we remark that (2.17) can also be proven directly by showing that $\mathbf{T}$ maps L^∞ to L^∞ and L^1 to weak L^1. Then the Marcinkiewicz interpolation theorem will imply (2.17).

Applying (2.17) to the function

$$f(r) = \begin{cases} |u(r,t)|r^{(n-1)/p}, & r \geq 0 \\ 0, & r < 0, \end{cases}$$

we have

$$\begin{aligned} &\int_0^{t+R} \left[\frac{1}{(t-\rho+R)^{(n-1)/2}} \int_\rho^{t+R} |u(r,t)|r^{(n-1)/p}(r-\rho)^{(n-3)/2} dr \right]^p d\rho \\ &\leq C \int_0^\infty |u(r,t)|^p r^{n-1} dr \\ &= C \int_{\mathbf{R}^n} |u(x,t)|^p dx. \end{aligned} \tag{2.18}$$

When $r \geq \rho$ and $1 < p \leq 2$, we observe that

$$r^{(n-1)/p} = r^{(n-1)/2}\ r^{(n-1)/p-(n-1)/2} \geq r^{(n-1)/2}\ \rho^{(n-1)/p-(n-1)/2}.$$

Hence (2.18) becomes

$$\begin{aligned} &\int_0^{t+R} \left[\frac{1}{(t-\rho+R)^{(n-1)/2}} \int_\rho^{t+R} |u(r,t)|r^{(n-1)/2}(r-\rho)^{(n-3)/2} dr \right]^p \rho^{n-1-(n-1)p/2} d\rho \\ &\leq C \int_{\mathbf{R}^n} |u(x,t)|^p dx. \end{aligned} \tag{2.19}$$

From (2.10) and the fact that *supp* $u(\cdot, t) \subset B(0, t+R)$, we know that

$$\begin{aligned}\mathbf{R}(|u|)(\rho,t) &= c_n \int_\rho^{t+R} |u(r,t)| r (r^2-\rho^2)^{(n-3)/2} dr \\ &\le c_n \int_\rho^{t+R} |u(r,t)| r (r+\rho)^{(n-3)/2} (r-\rho)^{(n-3)/2} dr \\ &\le c \int_\rho^{t+R} |u(r,t)| r^{(n-1)/2} (r-\rho)^{(n-3)/2} dr.\end{aligned} \tag{2.20}$$

Substituting (2.20) to (2.19), we reach

$$\int_0^{t+R} \frac{[\mathbf{R}(|u|)(\rho,t)]^p}{(t-\rho+R)^{(n-1)p/2}} \; \rho^{n-1-(n-1)p/2} d\rho \le C \int_{\mathbf{R}^n} |u(x,t)|^p dx. \tag{2.21}$$

Using the lower bound of $\mathbf{R}(|u|)$ in (2.15) and (2.21), we deduce

$$\int_{\mathbf{R}^n} |u(x,t)|^p dx \ge C \int_0^{t-R-1} \frac{(t-\rho-R)^{np-(n-1)p^2/2}}{(t-\rho+R)^{(n-1)p/2}} \; \rho^{n-1-(n-1)p/2} d\rho.$$

When $\rho \in (0, t-R-1)$, it is clear that there exists $c_R > 0$ such that, for all $t > 2(R+1)$,

$$t-\rho+R \le c_R (t-\rho-R).$$

Hence there exist $C_R > 0$ such that

$$\int_{\mathbf{R}^n} |u(x,t)|^p dx \ge C_R \int_0^{t-R-1} \frac{\rho^{n-1-(n-1)p/2}}{(t-\rho-R)^{(n-1)p/2-np+(n-1)p^2/2}} d\rho. \tag{2.22}$$

Recall that p is the critical exponent for (1.1), i.e.

$$(n-1)p^2 - (n+1)p - 2 = 0.$$

It follows that

$$(n-1)p/2 - np + (n-1)p^2/2 = \frac{(n-1)p^2-(n+1)p}{2} = 1.$$

Therefore (2.22) becomes

$$\int_{\mathbf{R}^n} |u(x,t)|^p dx \ge C_R \int_0^{t-R-1} \frac{\rho^{n-1-(n-1)p/2}}{(t-\rho-R)} d\rho. \tag{2.23}$$

Hence

$$\begin{aligned}&\int_{\mathbf{R}^n} |u(x,t)|^p dx \\ &\quad \ge C_R \int_{(t-R-1)/2}^{t-R-1} \frac{\rho^{n-1-(n-1)p/2}}{(t-\rho-R)} d\rho \\ &\quad \ge C_R (t-R-1)^{n-2-(n-1)p/2} \int_{(t-R-1)/2}^{t-R-1} \frac{1}{t-\rho-R} d\rho.\end{aligned}$$

So finally, we reach the refined lower bound

$$\int_{\mathbf{R}^n} |u(x,t)|^p dx \ge C (t-R)^{n-1-(n-1)p/2} \ln \frac{t-R}{2}. \tag{2.24}$$

From (2.2'), the above shows

$$\frac{d^2F_0(t)}{dt^2} = \int_{\mathbf{R}^n} |u(x,t)|^p dx \ge C(t-R)^{n-1-(n-1)p/2} \ln \frac{t-R}{2},$$

provided that t is sufficiently large. Comparing with the lower bound in (2.5′), the above contains an additional $\ln t$ term. This the key improvement.

Since $n-1-(n-1)p/2 \ge 0$ when $n \ge 4$, after integration we deduce, for large t,

$$F_0(t) \ge C(t-R)^{n+1-(n-1)p/2} \ln t.$$

Hence

$$F_0(t) \ge C(t+R)^{n+1-(n-1)p/2} \Big(\frac{t-R}{t+R}\Big)^{n+1-(n-1)p/2} \ln t,$$

when t is sufficiently large. Notice that

$$\lim_{t\to\infty} \Big(\frac{t-R}{t+R}\Big)^{n+1-(n-1)p/2} \ln t = \infty.$$

Therefore

$$F_0(t) \ge K_0(t+R)^{n+1-(n-1)p/2} \tag{2.25}$$

with $K_0 > 0$ being arbitrarily large when t is sufficiently large.

Also, recall from (2.3) that

$$\frac{d^2F_0(t)}{dt^2} \ge K_1(t+R)^{-n(p-1)}|F_0(t)|^p$$

with $K_1 = 1/(\mathrm{vol}(\mathbf{B}^n))^{p-1}$ being fixed.

This together with (2.25) and Lemma 2.1 with parameters

$$a \equiv n+1-(n-1)p/2 \quad \text{and} \quad q \equiv n(p-1)$$

imply Theorem 1.1 since $p = p_c$ satisfies

$$(p-1)(n+1-(n-1)p/2) = n(p-1) - 2 \quad \text{and} \quad p > 1.$$

This shows that all solutions of (1.1) with nontrivial nonnegative initial values must blow up in finite time. □

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